THE MAXIMUM-LIKELIHOOD ESTIMATE FOR CONTINGENCY TABLES WITH ZERO DIAGONAL

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The Maximum-Likelihood Estimate for Contingency Tables with Zero Diagonal

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Under a reasonable hypothesis about the distribution of trade shipments among a group of K countries, or of non-self-directed acts among any group of K individuals, the likelihood of obtaining a given table of observed frequencies n_{ij} is

$$L = \frac{N!}{\Pi n_{ij}!} \cdot \frac{P_1^{n_1} \cdots P_K^{n_K} \cdot Q_1^{m_1} \cdots Q_K^{m_K}}{[1 - P_1 \cdot Q_1 - \cdots - P_K \cdot Q_K]^N}$$

where $n_i = n_{i1} + \cdots + n_{iK}$ and $m_i = n_{1i} + \cdots + n_{Ki}$ are the total number of shipments made and received, respectively, by country i, N is the total number of shipments altogether, and P_i and Q_i are the theoretical tendencies of country i to ship and to receive shipment respectively and satisfy $\Sigma P_i = \Sigma Q_i = 1$. In the attempt to test the correctness of the hypothesis, a critical problem is that of finding the maximum-likelihood estimates of the P's and Q's, that is, the values making L as large as possible. We give all solutions and show that, for the usual table of observations, the solution is unique and amounts to finding the value of α for which $K - 2 - R_1 - \cdots - R_K$ is 0 —

where $R_i = \sqrt{\left(1 - \frac{m_i + n_i}{\alpha}\right)^2 - \frac{4m_i n_i}{\alpha^2}}$ and setting

$$P_i = (1/2)\left(1 + \frac{n_i - m_i}{\alpha} - R_i\right) \quad \text{and} \quad Q_i = (1/2)\left(1 + \frac{m_i - n_i}{\alpha} - R_i\right).$$

1. INTRODUCTION AND SUMMARY

In the analysis of social interactions, the problem arises [8] of maximizing the likelihood function

$$L = c \cdot \frac{P_1^{n_1} \cdots P_K^{n_K} \cdot Q_1^{m_1} \cdots Q_K^{m_K}}{[1 - P_1 \cdot Q_1 - \cdots - P_K \cdot Q_K]^N} \tag{1.1}$$

over all non-negative P_i and Q_i, subject to the restraints

$$P_1 + \cdots + P_K = 1 \quad \text{and} \quad Q_1 + \cdots + Q_K = 1. \tag{1.2}$$

Here $m_1, \cdots, m_K, n_1, \cdots, n_K$ $(K \geq 2)$ are given non-negative integers satisfying $n_1 + \cdots + n_K = m_1 + \cdots + m_K = N$, and c is a positive number. It is further given that, for all i, $n_i + m_i \leq N$.

Various (iterative) procedures for maximizing L have been suggested [1, 3, 4, 8]; however, for none of them is it known whether the resulting sequences

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converge.[1] In the following we give a procedure for finding all solutions, and show that, in the "general case," the solution is unique.

Briefly, the source of the problem is this. We suppose that there are K individuals directing acts of some kind toward one another and that, during some period of observation, they made a total of N such acts. The individuals may be countries, people, monkeys, or whatever, and the acts may be trade shipments, threats, hits, or whatever; however, for the sake of concreteness, we shall discuss the problem in terms of countries and shipments. Letting n_i and m_i be the number of shipments made and received, respectively, by country i, and n_{ij} the number sent by country i to country j, we have $n_i = n_{i1} + \cdots + n_{iK}$, $m_i = n_{1i} + \cdots + n_{Ki}$, and $n_1 + \cdots + n_K = m_1 + \cdots + m_K = N$. Such data are sometimes displayed in the form of a matrix (n_{ij}), whence the n_i's and m_i's may be called the row and column totals (or marginals) respectively. Since we are concerned only with exports and imports, the main diagonal will contain nothing but zeroes.

If the shipments are made independently of one another, that is, if at each trial (occurrence of a single shipment) there is a constant probability p_{ij} that it will be country i shipping to country j, and if the distribution of probabilities exhibits sender-receiver independence [8, p. 555; 5, p. 1093], that is, if the various ratios obtainable between the p_{ij}'s of one row are the same as the corresponding ratios in any other row—except where a diagonal element is involved—then there are two sets, $P_1, \cdots, P_K$ and $Q_1, \cdots, Q_K$, of non-negative "parameters" satisfying (1.2) and such that

$$p_{ij} = \frac{P_i \cdot Q_j}{t} \quad \text{for } i \neq j, \qquad p_{ij} = 0 \quad \text{for } i = j \tag{1.3}$$

where

$$t = \sum_{i \neq j} P_i \cdot Q_j = 1 - P_1 Q_1 - \cdots - P_K Q_K,$$

and the likelihood L of obtaining in N trials exactly the distribution observed is given by (1.1), where

$$c = \frac{N!}{n_{11}! n_{12}! \cdots n_{KK}!}.$$

The parameters P_i and Q_i may be called the "theoretical tendencies" of country i to ship and receive shipment respectively; they are, in general, only approximately equal to the true tendencies (probabilities) p_i and q_i, since

$$p_i = p_{i1} + \cdots + p_{iK} = \frac{P_i(1 - Q_i)}{t} \quad \text{and}$$
$$q_i = p_{1i} + \cdots + p_{Ki} = \frac{(1 - P_i)Q_i}{t}. \tag{1.4}$$

The conjunction of the two assumptions about independence, upon which the

[1] A referee has pointed out that in Section 4 of [5] a procedure is given which is convergent under the conditions specified there.

validity of (1.1) must depend, is called our "null-hypothesis" or "model." (For further details see [8].)

Typically, the P's and Q's are not known, but are rather to be estimated from the observed n_i's and m_i's. Then, after making such an estimation, one usually compares the predicted or expected values $E_{ij}=p_{ij}\cdot N$ with the observed entries n_{ij} to see whether the overall disagreement is great enough to discredit the null-hypothesis—for in most applications it is in no way given that the model is correct.

Of the several kinds of estimation possible, we are concerned only with the maximum-likelihood estimate, which consists of finding those P's and Q's making L a maximum. In the following we show how to find every such set of P's and Q's, when they exist, and explain the conditions under which they will fail to exist.

The reader interested in the practical results can refer to the flow chart (Figure 1) or to the following summary. (In both, multiple solutions have been eliminated, where they exist, either by demanding symmetry between the P and Q values or by setting equal to zero any parameter not required to be positive.) In addition, an illustrative example will be found at the end of the article (p. 1379).

Summary. The function L can be maximized according to the following schedule. For (1), (2) and (3), other solutions exist which are described in the text.

1. If all the n_{ij} are zero except n_{hk}, set $P_h=Q_k=1$, all others 0.
2. If all row-totals are zero except n_h but two or more column totals are non-zero, set $P_h=1$, $Q_h=0$, and, for $i\neq h$, $P_i=0$ and $Q_i=m_i/N$. If all column-totals are zero except m_k but two or more row totals are non-zero, set $P_k=0$, $Q_k=1$, and, for $i\neq k$, $P_i=n_i/N$ and $Q_i=0$.
3. If all the n_{ij} are zero except n_{hk} and n_{kh}, set

$$P_h = Q_k = \frac{n_h - \sqrt{n_h m_h}}{n_h - m_h}, \qquad P_k = Q_h = \frac{m_h - \sqrt{n_h m_h}}{m_h - n_h},$$

any others zero. (In case $n_h - m_h = 0$, set $P_h=P_k=Q_h=Q_k=1/2$.)
4. If no n_i or m_j or $n_{hk}+n_{kh}$ equals N, but all n_{ij} are zero except those in some one row and the corresponding column, there is no solution. (Gather more data.)
5. If no one row plus corresponding column contains all the non-zero n_{ij}, choose j_0 so that μ_{j_0} will be $\geq \mu_i = m_i+n_i+2\sqrt{m_i n_i}$ for all i, define $U(\alpha) = K-2-R_1-\cdots-R_K$ and $V(\alpha)=K-2+R_{j_0}-\Sigma_{i\neq j_0}R_i$ where

$$R_i = \sqrt{\left(1-\frac{m_i+n_i}{\alpha}\right)^2 - \frac{4m_in_i}{\alpha^2}}$$

and α is the unique solution to $U(\alpha)=0$ or $V(\alpha)=0$, and set

$$P_i = (1/2)\left(1+\frac{n_i-m_i}{\alpha} - R_i\right),$$

$$Q_i = (1/2)\left(1+\frac{m_i-n_i}{\alpha} - R_i\right)$$

for $i\neq j_0$, and

$$P_i = (1/2)\left(1+\frac{n_i-m_i}{\alpha} - CR_i\right),$$

$$Q_i = (1/2)\left(1+\frac{m_i-n_i}{\alpha} - CR_i\right)$$

for $i=j_0$ where $C=1$ if $U(\alpha)=0$ and $C=-1$ if $V(\alpha)=0$.

The remainder of the article is devoted to proving the foregoing results and justifying the procedures given in the flow chart.

2. PRELIMINARY REMARKS

To place the problem in an analytic setting let S denote the subset of E^{2K} (Euclidean $2K$-space) which contains the point $x=(P_1, \cdots, P_K, Q_1, \cdots, Q_K)$ only if $\Sigma P_iQ_i<1=\Sigma P_i=\Sigma Q_i$ with each P_i and $Q_i\geq 0$, and let R denote the set of all numbers; then L is a continuous function from S to R which is bounded from above by 1.

We notice immediately that S is not compact (since it fails to contain, for example, the point with $P_1=Q_1=1$, others 0), so it is not certain that L will in fact have a maximum. Of course it will have a least upper bound, say b, but there may be no point where L takes on the value b. That will depend on conditions which we shall presently describe.

3. SPECIAL CASES: MULTIPLE SOLUTIONS

In case $N=0$, L will equal 1 whatever the values of the P's and Q's. (In no trials one is certain to get no results.) In what follows we shall assume that $N\neq 0$.

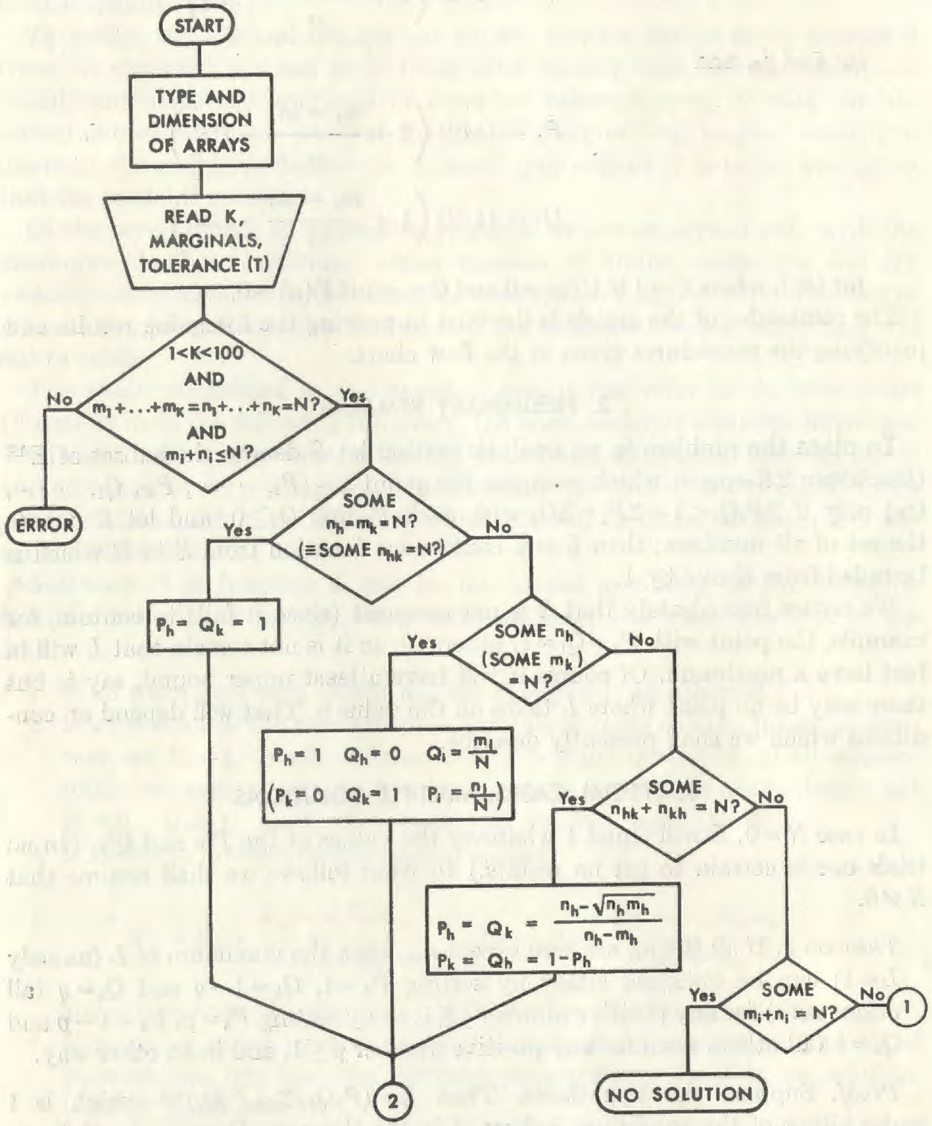
Theorem 1. If all the n_{ij} are zero except n_{hk}, then the maximum of L (namely $L=1$) can be obtained either by setting $P_h=1$, $Q_h=1-q$ and $Q_k=q$ (all others zero) for any positive number $q\leq 1$, or by setting $P_h=p$, $P_k=1-p$ and $Q_k=1$ (all others zero) for any positive number $p\leq 1$, and in no other way.

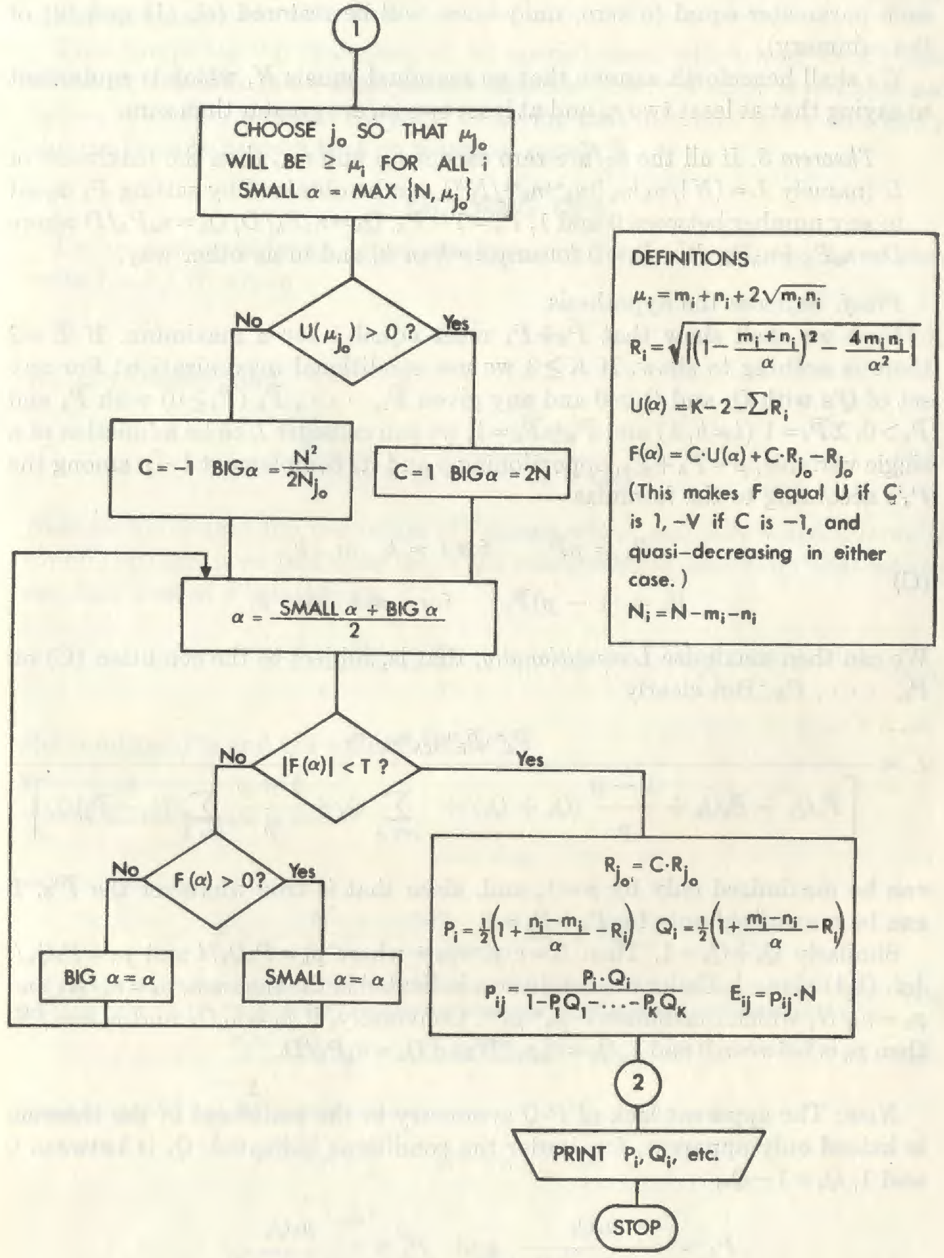
Proof. Suppose the hypothesis. Then $L=(P_hQ_k/\Sigma_{i\neq j}P_iQ_j)^{n_{hk}}$ which is 1 under either of the conditions indicated in the theorem. Conversely, if $L=1$, then P_h and $Q_k>0$, but $P_kQ_h=0$ and $P_i=Q_i=0$ for $i\neq h$, k—whence one of P_h and Q_k must equal 1.

Note: In the next two theorems, and in the "factoring of L," we make use of the fact that a product like $p_1^{n_1}\cdots p_K^{n_K}$ with $p_i\geq 0$ and $\Sigma p_i=1$, can be maximized only by $p_i=n_i/N$ for all i.

Theorem 2. If all row totals are zero except, say, n_1, but two or more column totals are non-zero, then the maximum of L (namely $L=(N!/m_2!\cdots m_K!)\cdot m_2^{m_2}\cdots m_K^{m_K}/N^N$) can be obtained by setting $P_1=1$, $Q_1=1-q$ and $Q_i=q\cdot m_i/N$ $(i=2, \cdots, K)$ for any positive number $q\leq 1$, and in no

Figure 1. FLOW CHART FOR MAXIMIZING L





other way. (And analogously for any other row total n_i or any column total m_j.)

Proof. Suppose the hypothesis. Then $m_1=0$ and, for any set of Q's,

$$L = c\cdot \frac{Q_2^{m_2}\cdots Q_K^{m_K}}{\left(\frac{1-P_1}{P_1}Q_1+\frac{1-P_2}{P_1}Q_2+\cdots+\frac{1-P_K}{P_1}Q_K\right)^N}$$

is maximal only for $P_1=1$, others 0. But then, writing $q=1-Q_1$ and $\overline{Q}_i=Q_i/q$ for $i=2,\cdots,K$, we have $L=c.\overline{Q}_2^{m_2}\cdots\overline{Q}_K^{m_K}$ which we know is maximal only if $\overline{Q}_i=m_i/N$ for all i.

Theorems 1 and 2 cover all the possibilities where some marginal equals N. We can perhaps make plausible the non-uniqueness of the solutions by observing that, if country 1 does all the shipping, there is no test of its theoretical receptivity, Q_1, and we can make it whatever we like. If we agree to set any such parameter equal to zero, uniqueness will be restored (cf. (1) and (2) of the summary).

We shall henceforth assume that no marginal equals N, which is equivalent to saying that at least two n_i and at least two m_i are greater than zero.

Theorem 3. If all the n_{ij} are zero except n_{hk} and n_{kh}, then the maximum of L (namely $L=(N!/n_h!n_k!)n_h^{n_h}n_k^{n_k}/N^N$) can be obtained by setting P_h equal to any number between 0 and 1, $P_k=1-P_h$, $Q_h=n_kP_h/D$, $Q_k=n_hP_k/D$ where $D=n_kP_h+n_hP_k$, $P_i=Q_i=0$ for any $i\neq h$ or k, and in no other way.

Proof. Suppose the hypothesis.

First we shall show that P_h+P_k must equal 1 for a maximum. If $K=2$ there is nothing to show. If $K\geq 3$ we use conditional maximization: For any set of Q's with Q_h and $Q_k>0$ and any given $\overline{P}_1, \cdots, \overline{P}_k$ ($\overline{P}_i\geq 0$) with $\overline{P}_h$ and $\overline{P}_k>0$, $\Sigma\overline{P}_i=1$ $(i\neq h, k)$ and $\overline{P}_h+\overline{P}_k=1$, we can consider L to be a function of a single variable, $p=P_h+P_k$, apportioning p and its complement $1-p$ among the P_i's according to the formulas

$$\text{(C)}\qquad \begin{aligned} P_i &= p\overline{P}_i \qquad \text{for } i = h \text{ or } k,\\ P_i &= (1-p)\overline{P}_i \qquad \text{for } i\neq h \text{ or } k. \end{aligned}$$

We can then maximize L *conditionally*, that is, subject to the condition (C) on $P_1, \cdots, P_K$. But clearly

$$L = \frac{\overline{P}_h^{n_h}\overline{P}_k^{n_k}Q_h^{m_h}Q_k^{m_k}}{\left[\overline{P}_hQ_k + \overline{P}_kQ_h + \dfrac{1-p}{p}(Q_h+Q_k) + \displaystyle\sum_{j\neq h,k} Q_j + \dfrac{1-p}{p}\sum_{j\neq h,k}(1-\overline{P}_j)Q_j\right]^N}$$

can be maximized only by $p=1$, and, since that is true whatever the $\overline{P}$'s, L can be maximized only by $P_h+P_k=1$.

Similarly $Q_h+Q_k=1$. Then $L=c\cdot p_h^{n_h}p_k^{n_k}$ where $p_h=P_hQ_k/t$ and $p_k=P_kQ_h/t$ [cf. (1.4) above]. Under the conditions indicated in the theorem, $p_h=n_h/N_h$ and $p_k=n_k/N$, which maximizes $c\cdot p_h^{n_h}p_k^{n_k}$. Conversely, if $p_h=n_h/N$ and $p_k=n_k/N$, then p_h is between 0 and 1, $Q_h=n_kp_h/D$ and $Q_k=n_hP_k/D$.

Note: The apparent lack of P-Q symmetry in the statement of the theorem is indeed only apparent, for, under the conditions indicated, Q_k is between 0 and 1, $Q_h=1-Q_k$,

$$P_k = \frac{n_hQ_k}{n_hQ_k+n_kQ_h} \quad\text{and}\quad P_h = \frac{n_kQ_h}{n_hQ_k+n_kQ_h}.$$

Theorems 1 and 3 cover all the possibilities where the observed trade involves only two countries, h and k. The non-uniqueness of the solutions can perhaps be understood by observing that, since country h can ship only to country k, a decrease in P_h can be compensated for by an increase in Q_k. If, on the other

hand, we require that P_h equal Q_k, uniqueness will be restored (cf. (1) and (3) of the summary).

This completes the discussion of all special cases which lead to multiple solutions. Henceforth we shall assume that the observed trade is not confined to two countries, which is the same as saying that $n_{ij}+n_{ji}<N$ for all i and j, and continue to assume that no marginal equals N.

4. FACTORING OF L

Let us now consider an alternate expression for L. If each $Q_i<1$, we may write $L=c\cdot f\cdot G$, where

$$f = p_1^{n_1}\cdots p_K^{n_K} \tag{4.1}$$

[cf. (1.4) earlier] and

$$G = \frac{Q_1^{m_1}\cdots Q_K^{m_K}}{(1-Q_1)^{n_1}\cdots(1-Q_K)^{n_K}}. \tag{4.2}$$

Now we know that the maximum of f occurs when, and only when, $p_i=n_i/N$ for all i. Hence, if we find some set of Q's maximizing G, and if, for that set, we can find a set of P's satisfying

$$\frac{P_i(1-Q_i)}{t} = \frac{n_i}{N} \quad \text{for } i = 1, \cdots, K \tag{4.3}$$

the resulting P's and Q's will maximize L.[2]

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[2] We note that L can also be written

$$L=c\cdot F\cdot g,$$

with

$$F = \frac{P_1^{n_1}\cdots P_K^{n_K}}{(1-P_1)^{m_1}\cdots(1-P_K)^{m_K}} \quad\text{and}\quad g = q_1^{m_1}\cdots q_K^{m_K},$$

and mention that Savage, Deutsch, Alker and Goodman have considered the possibility of maximizing f and g simultaneously—that is, the possibility of solving simultaneously the $2K$ equations in $2K$ unknowns

$$\frac{P_i(1-Q_i)}{t} = \frac{n_i}{N}, \qquad \frac{(1-P_i)Q_i}{t} = \frac{m_i}{N} \qquad i = 1, \cdots, K. \tag{4.3'}$$

They have proposed the following iterative procedure: set

$$P_i^{(1)} = p_i = \frac{n_i}{N}, \qquad Q_i^{(1)} = q_i = \frac{m_i}{N} \quad\text{and}$$

$$P_i^{(n+1)} = \frac{\dfrac{p_i}{1-Q_i^{(n)}}}{\displaystyle\sum_1^K \frac{p_j}{1-Q_j^{(n)}}} \quad\text{and}\quad Q_i^{(n+1)} = \frac{\dfrac{q_i}{1-P_i^{(n)}}}{\displaystyle\sum_1^K \frac{q_j}{1-P_j^{(n)}}}$$

for each positive integer n, and set $P_i=\lim_{n\to\infty} P_i^{(n)}$, $Q_i=\lim_{n\to\infty} Q_i^{(n)}$ if those limits exist.

On the assumption that the procedure is convergent (and the implicit assumption that each limit is <1), Goodman [3] has shown that it will yield a solution to (4.3′), and hence maximizes f and g. That it will also maximize F and G—and hence L—is, however, not immediately clear.

Theorem 4. For every set of Q's making $G>0$, there exists a unique set of P's satisfying (4.3); it is the solution to the system of simultaneous equations

$$\begin{array}{lllll} \left(1-Q_1+\frac{n_1}{N}Q_1\right)P_1+ & \frac{n_1}{N}Q_2P_2 & +\cdots+ & \frac{n_1}{N}Q_KP_K & =\frac{n_1}{N} \\ \frac{n_2}{N}Q_1P_1 & +\left(1-Q_2+\frac{n_2}{N}Q_2\right)P_2+ & \cdots+ & \frac{n_2}{N}Q_KP_K & =\frac{n_2}{N} \\ & \vdots & & & \\ \frac{n_K}{N}Q_1P_1 & +\frac{n_K}{N}Q_2P_2 & +\cdots+ & \left(1-Q_K+\frac{n_K}{N}Q_K\right)P_K & =\frac{n_K}{N} \end{array} \tag{4.4}$$

and it will have the properties that $P_1+\cdots+P_K=1$ and $P_i \geq 0$ for all i.

Proof. Suppose that $Q_1, \cdots, Q_K$ make $G>0$. Since no $Q_i=1$, the matrix of the system (4.4) has a strictly dominant main diagonal [9, p. 13]; whence (4.4) has a unique solution $P_1, \cdots, P_K$. Adding the equations of (4.4), we have $P_1+\cdots+P_K=1$. Now $t>0$ [since some $P_i>0$ and $(1-Q_i)P_i=(n_i/N)(1-P_1Q_1-\cdots-P_KQ_K)=(n_i/N)t$], whence $P_1, \cdots, P_K$ satisfy (4.3) and each P_i is ≥ 0.

Since a system like (4.4) may readily be solved by any one of several standard methods, we may confine our attention to the maximizing of G. As a matter of fact, however, in both of the remaining cases we shall find ways of calculating the P's without having to deal with (4.4).

The domain of G is not compact; hence G may fail to have a maximum. On the other hand, from Theorem 4 we see that, if L has a maximum, so does G—namely max $L/(c \cdot n_1^{n_1} \cdots n_K^{n_K}/N^N)$, and that, if some set of P's and Q's maximizes L, the Q's will maximize G. The precise conditions under which G and L will have or fail to have a maximum are given in Theorems 5 and 7—for the sake of which we introduce the abbreviation $M_i=m_i+n_i$.

5. "CLEARING-HOUSE" CASE; NO SOLUTION

Theorem 5. If N equals, say, M_1, then L has no maximum. However, the least upper bound of L, namely

$$b = c \cdot \frac{n_2^{n_2} \cdots n_K^{n_K} \cdot m_2^{m_2} \cdots m_K^{m_K}}{N^N}$$

can be approximated as closely as one likes by approaching the "singular point," $P_1=1$, $Q_1=1$, sufficiently closely and in the right way—that is, by taking a sufficiently small $q>0$, setting $Q_1=1-q$, $Q_i=q(m_i/n_1)$ for $i=2, \cdots, K$, and determining the corresponding P's through (4.4). (And analogously for any other $M_i=N$.)

Proof. Suppose that $M_1=N$. For any given $\bar{Q}_2, \cdots, \bar{Q}_K$ with sum 1 making $D=\bar{Q}_2^{m_2} \cdots \bar{Q}_K^{m_K}$ positive, we can attempt to maximize G conditionally over all $Q_1<1$, with $Q_2, \cdots, Q_K$ defined by $Q_i=q \cdot \bar{Q}_i$.

But

$$\begin{aligned} G &= \frac{Q_1^{m_1} \cdot q^{N-m_1} \cdot \bar{Q}_2^{m_2} \cdots \bar{Q}_K^{m_K}}{(1-Q_1)^{n_1} \cdot (1-Q_2)^{n_2} \cdots (1-Q_K)^{n_K}} \\ &= \frac{Q_1^{n_2}}{(1-q\bar{Q}_2)^{n_2}} \cdots \frac{Q_1^{n_K}}{(1-q\bar{Q}_K)^{n_K}} \cdot D \end{aligned} \tag{5.1}$$

can always be increased by increasing Q_1; and, since that is true whatever the $\bar{Q}_i$, G can have no maximum. However, since $\bar{Q}_i=m_i/n_1$ maximizes D, the l.u.b. of G is

$$\frac{m_2^{m_2} \cdots m_K^{m_K}}{n_1^{n_1}},$$

and the rest of the theorem follows.

In "determining the corresponding P's through (4.4)" we observe that $P_1 \to 1$ as $Q_1 \to 1$. Since, moreover, the condition $N=M_1$ is symmetric in m_1 and n_1, we might wonder if we can approximate b by choosing a sufficiently small $q>0$, setting $P_1=Q_1=1-q$, $P_i=q(n_i/m_1)$ and $Q_i=q(m_i/n_1)$ for $i=2, \cdots, K$—and thereby dispense with the routine but tedious solution of (4.4). We would then have

$$\lim_{Q_1 \to 0} L = c \cdot n_2^{n_2} \cdots n_K^{n_K} m_2^{m_2} \cdots m_K^{m_K}/2^N n_1^{n_1} m_1^{m_1}.$$

But, unfortunately, that is less than b if $n_1 \neq m_1$ (proof omitted). Thus, unless $n_1=m_1$, it is fruitless to approach the singular point along the line $P_1=Q_1$. The correct strategem is given in Theorem 6.

Theorem 6. If $N=M_1$ then b can be approximated as closely as desired by choosing a sufficiently small positive number ϵ and setting $P_1=1-\epsilon m_1$, $Q_1=1-\epsilon n_1$, and $P_i=\epsilon n_i$, $Q_i=\epsilon m_i$ for $i \neq 1$. (And analogously for any other $M_i=N$.)

(Proof omitted.)

Notice that, since $N_1=N-M_1$ is the sum of all the entries n_{ij} not in the first row or the first column, Theorems 5 and 6 cover the case where every such entry is zero, that is, where country 1 acts as a sort of clearinghouse for the others: nobody else can ship except to it nor receive shipment except from it. Of course, if we believed that that relation held for the parent population of shipments and not just for this sample, we would use a different model: $p_{i1}=p_i$ and $p_{1i}=q_i$ for $i \neq 1$, $p_{11}=0$ and $p_{ij}=0$ for $i, j \neq 1$, where $p_i=n_i/N$ and $q_i=m_i/N$, and hence $p_2+\cdots+p_K+q_2+\cdots+q_K=1$. (These p_{ij}'s are the limiting values of the probabilities obtained in accordance with (1.3) as we approach the singular point in accordance wlth Theorem 5 or Theorem 6.) And analogously for any other $N_i=0$.

6. THE "GENERAL" CASE: THERE IS A SOLUTION

If, on the other hand, $N_1>0$, we see from (5.1) that $\lim_{Q_1\to 0} G=0$, and analogously for any other $N_i>0$. Hence, if no $N_i=0$, there is a continuous extension of G to the (compact) closure of its present domain, the extension having the value zero at all the new points. Since the extension has a maximum, and since no such maximum occurs at any of the new points, we see that G itself must have a maximum. Thus we have

Theorem 7. If no $N_i=0$, L has a maximum.

Henceforth we shall assume that no $N_i=0$, or, what is the same thing, that $M_i<N$ for all i. That is what some people might call "the general case," since the other cases are rarely encountered.

While Theorem 7 is no doubt of some theoretical interest, it is scientifically useless. What is needed instead, and what in fact we develop in the remainder of the article, is a practical method for generating a maximum. In the process we shall establish the other important theoretical result, that the maximum of L is unique. The results are summarized in Theorems 11 and 12.

7. NECESSARY CONDITIONS; ELIMINATING EXTRANEOUS SOLUTIONS

Let us now find some necessary conditions for maximizing G. In the following, a "point" will always be a point $(Q_1, \cdots, Q_K)$ of E^K. From (5.1) we see that maximizing G requires $Q_1=0$ if $m_1=0$ and $Q_1>0$ if $m_1\neq 0$. In the latter case, the derivative

$$\frac{dG}{dQ_1} = \frac{m_1}{Q_1}G - \frac{N_1}{q}G - \frac{n_2\overline{Q}_2}{1-q\overline{Q}_2}G - \cdots - \frac{n_K\overline{Q}_K}{1-q\overline{Q}_K}G$$

$$= \frac{G}{qQ_1}\left[m_1 - Q_1\cdot\left(\frac{n_2}{1-Q_2}+\cdots+\frac{n_K}{1-Q_K}\right)\right]$$

must be zero. In fact, the quantity inside the brackets must be zero in either case. Thus, among the points satisfying the equations

$$m_1 - Q_1\cdot\left(\frac{n_2}{1-Q_2}+\frac{n_3}{1-Q_3}+\cdots+\frac{n_K}{1-Q_K}\right)=0$$

$$m_2 - Q_2\cdot\left(\frac{n_1}{1-Q_1}+\frac{n_3}{1-Q_3}+\cdots+\frac{n_K}{1-Q_K}\right)=0 \qquad (7.1)$$

$$m_K - Q_K\cdot\left(\frac{n_1}{1-Q_1}+\frac{n_2}{1-Q_2}+\cdots+\frac{n_{K-1}}{1-Q_{K-1}}\right)=0$$

will be found every point which maximizes G.

There may also be "extraneous solutions," (i.e., with some Q_i negative or with $Q_1+\cdots+Q_K\neq 1$): in the example

0	1	2	3
1	0	2	3
1	1	0	2
2	2	4	8

the equations are satisfied both by (1/4, 1/4, 1/2) and by (−2, −2, 2). We shall shortly give a simple condition—(A) of Theorem 8—to ensure that every Q_i is non-negative, and we remark now that any point satisfying (7.1) with

$$\alpha = \frac{n_1}{1-Q_1}+\cdots+\frac{n_K}{1-Q_K}$$

different from zero will also satisfy $\Sigma Q_i=1$. (Proof omitted.)

[*Parenthetical remark.* There is not much hope of solving (7.1) by algebraic manipulations alone. In case $K=3$, the problem can be reduced, after much calculation, to that of solving the quartic equations

$$a_iQ_i^4 + b_iQ_i^3 + c_iQ_i^2 + d_iQ_i + e_i = 0 \quad (i=1,2,3)$$

(where

$$a_1 = (n_2n_3 - m_2m_3)(m_1n_2n_3 + m_2n_1n_3 + m_3n_1n_2 - 2n_1n_2n_3 - m_1m_2m_3)$$

$$b_1 = n_1(n_2N_2 + n_3N_3)(3n_2n_3 - m_2m_3 - m_2n_3 - m_3n_2)$$
$$- 2m_1(n_2N_2 + n_3N_3)(n_2n_3 - m_2m_3) - n_1^2N_1(m_2 - n_2)(m_3 - n_3)$$

$$c_1 = -n_1^2N_1N_2(m_2 - n_2) - n_1^2N_1N_3(m_3 - n_3) + n_1N_2N_3(m_2n_3 + m_3n_2)$$
$$+ (m_1 - n_1)(n_2^2N_2^2 + n_2^2N_2^2 + 4n_2n_3N_2N_3) - 2m_1m_2m_3N_2N_3$$

$$d_1 = N_2N_3(n_1 - 2m_1)(n_2N_2 + n_3N_3) - n_1^2N_1N_2N_3$$

$$e_1 = m_1N_2^2N_3^2$$

and $a_2, \cdots, e_2, a_3, \cdots, e_3$ may be found by analogy). For $K>3$, the calculations become prohibitively lengthy, and we must look for another approach.]

If we introduce the abbreviation

$$\alpha = \frac{n_1}{1-Q_1}+\cdots+\frac{n_K}{1-Q_K} \qquad (7.2)$$

then (7.1) becomes

$$m_i + \frac{n_iQ_i}{1-Q_i} - Q_i\alpha = 0 \qquad (i=1,\cdots,K) \qquad (7.3)$$

or, after multiplication by $1-Q_i$,

$$\alpha Q_i^2 - (\alpha + m_i - n_i)Q_i + m_i = 0 \qquad (i=1,\cdots,K) \qquad (7.4)$$

among whose solutions must, as before, be found every point which maximizes G.[3]

The next theorem gives a condition to be added to (7.1) or (7.4) to eliminate extraneous solutions (points not in the domain of G).

Theorem 8. In order for a solution of (7.1)—or (7.4)—to have every $Q_i \geq 0$ it is necessary and sufficient that

$$\text{(A)} \qquad \frac{n_1}{1-Q_1} + \cdots + \frac{n_K}{1-Q_K} \geq N \cdot$$

Proof. Necessity of (A) is obvious, and sufficiency follows from (7.4):

$$Q_i = \frac{b_i \pm \sqrt{b_i^2 - 4m_i\alpha}}{2\alpha}$$

where $b_i = \alpha + m_i - n_i$ is positive if $\alpha > n_i$.

And, as already remarked, $\alpha \neq 0$ is sufficient to ensure $\Sigma Q_i = 1$. Thus, among the one or more points satisfying the conjunction (7.1A) of (7.1) and (A) must be found every point which maximizes G. And equally for the conjunction (7.4A) of (7.4) and (A).

8. THE NECESSARY CONDITIONS IN TERMS OF THE PARAMETER α

In (7.2) we have α expressed in terms of the Q's. It is more fruitful to think of α as the "independent variable" and use (7.4) to express the Q's in terms of α:

$$Q_i = \frac{b_i \pm r_i}{2\alpha} \qquad (i = 1, \cdots, K) \tag{8.1}$$

where $b_i = \alpha + m_i - n_i$ and

$$r_i = \sqrt{b_i^2 - 4m_i\alpha} = \sqrt{(\alpha - M_i)^2 - 4m_i n_i}.$$

Then we can say that the problem of solving (7.1A) is equivalent to the problem of finding every *usable* value of α—i.e., $\geq \mu_i = M_i + 2\sqrt{n_i m_i}$ for all i (so that $r_i = \sqrt{\alpha - \mu_i}\,\sqrt{\alpha - M_i + 2\sqrt{n_i m_i}}$ will be real) and $\geq N$—for which the Q's—as given by (8.1)—will satisfy (7.2). As a matter of fact, it suffices to find every value of α for which the corresponding Q's have sum 1:

Theorem 9. If, for some usable α, the Q's—as defined by (8.1)—satisfy $Q_1 + \cdots + Q_K = 1$, then

$$\frac{n_1}{1-Q_1} + \cdots + \frac{n_K}{1-Q_K} = \alpha.$$

(Proof omitted.)

In Theorem 9 we begin to see the rudiments of a procedure for maximizing G.

[3] The referee has kindly pointed out that Equation (7.4) was presented earlier by Blumen, Kogen, and McCarthy in [2] and by Goodman in [6], and that, in addition, methods for solving Equation 7.4 were discussed in Goodman's 1961 article [6], and the relationship between Equation 4.3′ (Footnote 2) and Equation 7.4 was discussed in Goodman's 1963 article [3, Section 4].

9. CORRECT SIGN FOR Q_i; THE FUNCTIONS U AND V

Now consider the choice of "sign" in the expressions (8.1) for $Q_1, \cdots, Q_K$. Since there are 2^K ways to distribute + and − among them, one might anticipate 2^K quests for an α to make $Q_1 + \cdots + Q_K = 1$. How practical that would be can perhaps be seen from the following consideration: supposing that each quest takes ten seconds, we calculate that, if $K = 20$, the whole process will take more than one year, while if $K = 40$, the process will take more than a million years. We therefore have to reduce the number of distributions that need to be considered.

Since $\Sigma Q_i = (1/2)(K \pm R_1 \pm \cdots \pm R_K)$—where $R_i = r_i/\alpha$—we shall need to use enough minus signs to make $\pm R_1 \pm \cdots \pm R_K = -(K-2)$. In fact, we cannot use more than one plus sign:

Lemma 1. For any usable α,

$$R_1 + R_2 - R_3 - \cdots - R_K > -(K-2)$$

(and equally for R_1, R_3 or any other choice of "plus terms").

Proof. For any usable α, $R_i \leq 1 - M_i/\alpha$, and hence $-R_3 - \cdots - R_K > -(K-2)$.

We can reduce the possibilities still further:

Lemma 2. If $\mu_1 \leq \mu_2$ (where $\mu_i = m_i + n_i + 2\sqrt{m_i n_i}$), then

$$R_1 - R_2 - \cdots - R_K > -(K-2),$$

(and equally for $\mu_1 \leq$ any other μ_j). And analogously for any $\mu_i \leq \mu_j$.

Proof. Suppose that $\mu_1 \leq \mu_2$. Since $R_1 - R_2 - \cdots - R_K \geq Z(\alpha) - (K-2)$, where $Z(\alpha) = R_1 - R_2 + (N_1 + N_2)/\alpha$, it suffices to show that $Z(\alpha) > 0$. If $\alpha = \mu_2$ then $R_2 = 0$ and there is nothing left to prove; so suppose that $\alpha > \mu_2$. Then it suffices to show that $z(\alpha) = \alpha(r_1 + r_2) \cdot Z(\alpha)$ is > 0.

Case 1. Suppose that $M_1 \leq M_2$. Then

$$z(\alpha) = (M_2 - M_1)(2\alpha - M_2 - M_1) + 4m_2n_2 - 4m_1n_1 + (N_1 + N_2)(r_1 + r_2)$$

is positive at $\alpha = \mu_2$ and increasing thereafter.

Case 2. Suppose that $M_1 > M_2$. For the sake of convenience, let

$$\beta = 2\frac{m_2n_2}{M_1} - \frac{m_1n_1}{M_2}, \qquad \gamma = \frac{M_1 + M_2}{2}, \qquad \delta = \frac{M_1 - M_2}{2},$$

and $a = \alpha - \beta - \gamma$. Notice that, since $4m_2n_2 - 4m_1n_1 - (M_1 - M_2)^2 \geq 4\sqrt{m_1n_1}\,(M_1 - M_2)$, we have $\beta - \delta \geq 2\sqrt{m_1n_1}$. Similarly, $\beta + \delta \geq 2\sqrt{m_2n_2}$. But then

$$\begin{aligned}\frac{z(\alpha)}{M_1 - M_2} &> r_1 + r_2 + \frac{1}{M_1 - M_2}(r_1^2 - r_2^2) \\ &= \sqrt{a^2 + 2a(\beta - \delta) + (\beta - \delta)^2 - 4m_1n_1} \\ &\quad + \sqrt{a^2 + 2a(\beta + \delta) + (\beta + \delta)^2 - 4m_2n_2} - 2a \\ &\geq 0\end{aligned}$$

whether a is positive, negative or zero.

Thus we must use all minus signs except, perhaps, with the R_i corresponding to the largest μ_i. That is, if j_0 is chosen so that $\mu = \mu_{j_0} = \max\{\mu_1, \cdots, \mu_K\}$ and U and V denote, respectively, the functions such that $U(\alpha) = K - 2 - R_1 - \cdots - R_K$ and $V(\alpha) = K - 2 + R_{j_0} - \Sigma_{i \neq j_0} R_i$ for all $\alpha \geq \mu$, we have

Theorem 10. The problem of solving (7.1A) is equivalent to that of finding every $\alpha \geq \mu$ such that

$$U(\alpha) = 0 \quad or \quad V(\alpha) = 0. \tag{9.1}$$

Theorem 10 will form the basis for a routine to maximize G.

10. THE "GENERAL" CASE (CONT'D.); THE SOLUTION IS UNIQUE

Since

$$U'(\alpha) = -\frac{1}{\alpha} \sum \left(\frac{\alpha - M_i}{r_i} - \frac{r_i}{\alpha} \right) < 0$$

for $\alpha > \mu$, U is strictly decreasing and hence can have no more than one zero. Moreover, since

$$U(\alpha) \xrightarrow{(\alpha \to \infty)} -2,$$

U will have a zero only if $U(\mu) \geq 0$.

To obtain an analogous result for V, we need two intermediate lemmas.

Lemma 3. Suppose that v is a continuous function on the stretch of numbers $[\mu, \infty)$. The following three statements are equivalent:

1. If v is zero at some number α_1 then there exists an $\alpha_2 > \alpha_1$ and (unless $\alpha_1 = \mu$) an $\alpha_0 < \alpha_1$ such that $v(\alpha)$ is positive for $\alpha_1 < \alpha < \alpha_2$ and negative for $\alpha_0 < \alpha < \alpha_1$.
2. If v is zero at some number α_1 then $v(\alpha)$ is positive for $\alpha > \alpha_1$ and (unless $\alpha_1 = \mu$) negative for $\mu \leq \alpha < \alpha_1$.
3. If v is zero at some number α_1 then there exists a positive function p on $[\mu, \infty)$ such that the product $y = p \cdot v$ has positive slope at α_1.

Proof. Part I. Suppose that (1) is true. If v has no zero or only one, then (2) is true; so suppose that v has at least two zeros. Since v is continuous it has a first such, say α_1, and since, by (1), α_1 is not a limit of other zeros, v has a next such, say α_1'. But then, by (1), v must be positive immediately to the right of α_1 and negative immediately to the left of α_1', which is impossible since there is no zero between α_1 and α_1'. Thus (1) implies (2).

Part II. Suppose that (2) is true. Suppose that v is zero at the number α_1 and let $p(\alpha) = (\alpha - \alpha_1)/v(\alpha)$ for $\alpha \neq \alpha_1$, $p(\alpha_1) = 1$. Clearly p is everywhere positive, and y defined by $y(\alpha) = p(\alpha) \cdot v(\alpha) = \alpha - \alpha_1$ has slope 1 everywhere. Thus (2) implies (3).

Part III. Suppose that (3) is true. Suppose that v is zero at the number α_1 and let p be a positive function such that $y = p \cdot v$ has positive slope at α_1. Then $y(\alpha_1) = 0$ and there exists a number $\alpha_2 > \alpha_1$ and (unless $\alpha_1 = \mu$) a number $\alpha_0 < \alpha_1$ such that y is positive between α_1 and α_2 and negative between α_0 and α_1. But then $v(\alpha) = y(\alpha)/p(\alpha)$ is positive for $\alpha_1 < \alpha_2 < \alpha_2$ and (unless $\alpha_1 = \mu$) negative for $\alpha_0 < \alpha < \alpha_1$. Thus (3) implies (1).

A continuous function satisfying (1), (2), (3) will be called quasi-increasing if it is zero somewhere, quasi-constant if it is everywhere positive or everywhere negative, and quasi-non-decreasing in either case.

Lemma 4. If $\mu_1 = \mu_{j_0}$ and v denotes the function defined by

$$v(\alpha) = \alpha \cdot V(\alpha) = (K - 2)\alpha + r_1 - r_2 - \cdots - r_K$$

for all $\alpha \geq \mu_1$, v is quasi-non-decreasing. And analogously for any other $\mu_i = \mu_{j_0}$.

Proof. Suppose the hypothesis. If v has no zero, it is quasi-constant, so suppose that v is zero somewhere. Then $m_1 n_1 > 0$—for otherwise $v(\alpha) \geq 2N_1 + r_1 - (\alpha - M_1)$ would be positive.

Case 1. Suppose that every zero of v is $> \mu_1$. Let α_1 be a zero of v. Without essential loss of generality we may suppose that, for all $i > 1$, $r_2 \leq r_i$ at $\alpha = \alpha_1$. We notice that, for all i, $r_1 \leq r_i$ at $\alpha = \alpha_1$—for otherwise $v(\alpha_1)$ would be positive. Since $\mu_1 >$ every other μ_i by Lemma 2, the function p, defined by $p(\alpha) = r_1 + r_2$ for all $\alpha \geq \mu_1$, is everywhere positive. Letting $y = p \cdot v$, we have

$$y(\alpha) = (r_1 + r_2 - 2\alpha)N_2 + (r_1 + r_2 + 2\alpha)N_1 + A + (r_1 + r_2)B$$

where

$$A = M_1^2 - M_2^2 - 4m_1n_1 + 4m_2n_2$$

and

$$B = \sum_3^K (\alpha - M_i - r_i).$$

But then, since

$$r_i' = \frac{dr_i}{d\alpha} = \frac{1}{\sqrt{1 - \dfrac{4m_in_i}{(\alpha - M_i)^2}}}$$

is ≥ 1 for $\alpha > \mu_1$, and

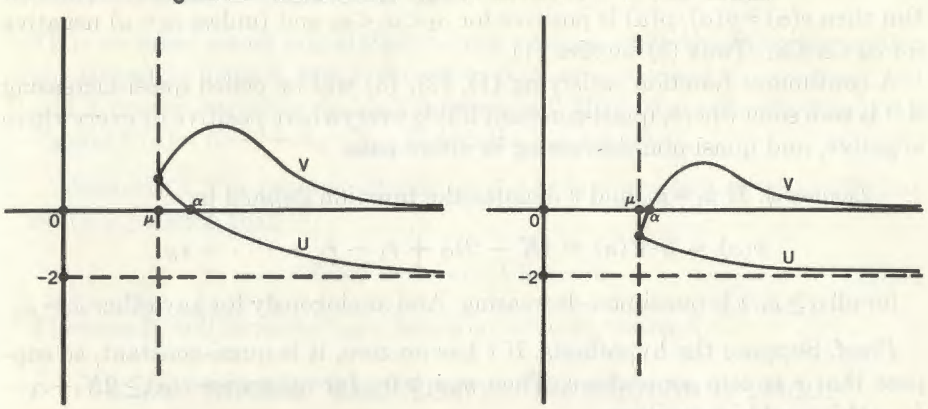
$$y'(\alpha) \geq 0 \cdot N_2 + 4N_1 + 2B + (r_1 + r_2) \sum (1 - r_i')$$
$$= 4N_1 + \sum \left(2 - \frac{r_1 + r_2}{r_i} \right) (\alpha - M_i - r_i)$$

for $\alpha > \mu_1$, $y(\alpha_1)$ is positive. Thus, in Case 1, v satisfies (3).

Case 2. Suppose no zero of v is $> \mu_1$. Then $\alpha_1 = \mu_1$ is the only zero of v and, since $m_1 n_1 > 0$, $\lim_{\alpha \to \mu_1} r_1' = +\infty$ and v satisfies (1).

Case 3. Suppose $v(\mu_1) = 0$ and v has a zero $> \mu_1$. But then, by the foregoing arguments, v is positive immediately to the right of μ_1 and negative immediately to the left of the first zero after μ_1, which is impossible.

Figure 2. LOCATION OF α IN THE TWO SUB-CASES



Corollary. V is quasi-non-decreasing.

Hence V has no zero if $V(\mu_{j_0})>0$ and at most one zero if $V(\mu_{j_0})<0$. In fact, in the latter case, V has exactly one zero by Theorem 7 since U has none (see Figure 2). Thus we have

Theorem 11. If $U(\mu_{j_0})>0$, the maximum of G may be obtained by setting

$$Q_i=\left(\frac{1}{2}\right)\left(1+\frac{m_i-n_i}{\alpha}-R_i\right) \qquad (i=1,\cdots,K)$$

where α is the unique solution to $U(\alpha)=0$—and in no other way.

Theorem 12. If $U(\mu_{j_0})\leq 0$, the maximum of G may be obtained by setting

$$Q_i=\left(\frac{1}{2}\right)\left(1+\frac{m_i-n_i}{\alpha}-R_i\right) \qquad \textit{for } i\neq j_0 \quad \textit{and}$$

$$Q_i=\left(\frac{1}{2}\right)\left(1+\frac{m_i-n_i}{\alpha}+R_i\right) \qquad \textit{for } i=j_0$$

where α is the unique solution to $V(\alpha)=0$—and in no other way.

This completes the theoretical solution of the problem of maximizing L. Since $U(\alpha)\geq -1+(N+N_{j_0})/\alpha$ is positive whenever $\alpha<N+N_{j_0}$, the case $U(\mu_{j_0})>0$ will no doubt be encountered more frequently than the case $U(\mu_{j_0})\leq 0$. In fact, the latter case results in a sum $P_{j_0}+Q_{j_0}\geq 1$, and indicates that country j_0 is especially active as sender and receiver.

The final theorem in this section shows that, although the equations (4.4) were needed to establish the feasibility of maximizing L by first maximizing G, in actual practice they need not be used at all; instead, one may find the P's in exactly the same way as the Q's.

Theorem 13. If $Q_1,\cdots,Q_K$ are defined by the appropriate formulas from Theorem 11 or Theorem 12, and $P_1,\cdots,P_K$ are defined by the corresponding formulas with m_i and n_i interchanged, then the P's and Q's will satisfy (4.4) and hence maximize f.

(Proof omitted.)

11. A PRACTICAL CONSIDERATION: BRACKETING α

In order to program a computer to approximate the unique α given by Theorems 11 and 12, it is useful to have an upper bound for α. That is, in the first case an α certain to make $U(\alpha)$ non-positive and in the second an α certain to make $V(\alpha)$ non-negative.

Theorem 14. If $\alpha\geq 2N$, $U(\alpha)<0$.

Proof. For then, writing $\rho=1-2M_i/\alpha$, we have

$$R_i=\sqrt{\frac{1}{\alpha^2}\left[M_i(2\alpha-3M_i)-4m_in_i\right]+\rho^2}$$

$$\geq\sqrt{\frac{1}{\alpha^2}\left[M_i(4M_i-3M_i)-4m_in_i\right]+\rho^2}$$

$$\geq\rho$$

(with equality only if $M_i=0$); whence $U(\alpha)<-2+4N/\alpha\leq 0$.

Theorem 15. If $\alpha\geq N^2/2N_{j_0}$, $V(\alpha)>0$.

Proof. For then, writing $\sigma=1-(N/\alpha)$, we have

$$R_{j_0}=\sqrt{\frac{1}{\alpha^2}\left[2\alpha N_{j_0}-N^2+(m_{j_0}-n_{j_0})^2\right]+\sigma^2}$$

$$\geq\sigma,$$

whence

$$V(\alpha)\geq\sigma-1+\frac{1}{\alpha}\sum_{i\neq j_0}M_i=\frac{N_{j_0}}{\alpha}>0.$$

Thus in the first case α may be sought in the interval $[\mu_{j_0}, 2N]$ and in the second in the interval $[\mu_{j_0}, N^2/2N_{j_0}]$. (If $\mu_{j_0}<N$, the first interval can be reduced to $[N, 2N]$.) We can then approximate α as closely as we like by successively halving the appropriate interval (or by any other suitable technique).

Finally we note that, although we have not discussed cases in which the "senders" may be different from the "receivers," that is, where the country or individual corresponding to the ith row is not necessarily the same as that corresponding to the ith column, our analysis applies to those cases as well. For example, the rows might represent all the married men at some social gathering, and the columns their wives, with, say, n_{ij} the number of times that male i talks to female j. Then, if we neglect the number of times that a man talks to his own wife, we again have a zero diagonal and can proceed as before.

12. AN EXAMPLE

In [7, p. 157], Ploog gives the following figures concerning the distribution of genital displays among three squirrel monkeys, R, S and U. (The genital display is a certain stereotyped signal peculiar to the species.)

		RECEIVER			
		R	S	U	
SENDER	R	0	1	8	9
	S	29	0	46	75
	U	2	3	0	5
		31	4	54	89

That is, during some period of observation, R (monkey 1) made one display to S and eight displays to U—a total of nine displays—while receiving 29 displays from S and two from U—a total of 31 displays received; and similarly for the others. We want to know if those figures are consistent with the hypothesis that each monkey apportions his displays among the other monkeys entirely on the basis of their respective theoretical tendencies to receive displays—the hypothesis of sender-receiver independence—or if, on the contrary, some monkey shows, so to speak, some favoritism in distributing his displays.

To answer that question, we must first calculate the value of the parameter α, use that value to compute the theoretical tendencies P_i and Q_i, use the P's and Q's in turn to derive expected values, and finally compare those values with the observed frequencies to see whether the overall disagreement is great enough to cause rejection of the hypothesis of sender-receiver independence.

The work may be laid out as follows. We calculate the values

$$M_1 = m_1 + n_1 = 40 \qquad M_2 = m_2 + n_2 = 79 \qquad M_3 = m_3 + n_3 = 59$$

$$4m_1n_1 = 1116 \qquad 4m_2n_2 = 1080 \qquad 4m_3n_3 = 1200$$

$$\mu_1 = 40 + \sqrt{1116} \qquad \mu_2 = 79 + \sqrt{1080} \qquad \mu_3 = 59 + \sqrt{1200}$$

$$< 40 + 34 < \mu_2 \qquad = 113.641016151 \qquad < 59 + 35 < \mu_2$$

and then carry out (see table) the iterative procedure, given in the flow chart, to approximate the α making $R_1 \pm R_2 + R_3$ equal to $K-2$ $(=1)$, where

$$R_1 = \sqrt{\left(1 - \frac{40}{\alpha}\right)^2 - \frac{1116}{\alpha^2}}, \qquad R_2 = \sqrt{\left(1 - \frac{79}{\alpha}\right)^2 - \frac{1200}{\alpha^2}},$$

$$R_3 = \sqrt{\left(1 - \frac{59}{\alpha}\right)^2 - \frac{1080}{\alpha^2}}.$$

RESULTS OF THE ITERATIVE PROCEDURE

	α	R_1	R_2	R_3	ΣR_i
1	113.641 016 151	.577...	0	.384...	.961...
2	145.820 508 075	.688...	.391...	.551...	1.631...
3	129.730 762 113	.641...	.285...	.482...	1.410...
4	121.685 889 132	.612...	.204...	.438...	1.256...
5	117.663 452 642	.	.	.	1.154...
6	115.652 234 396	.	.	.	1.089...
7	114.646 625 274	.	.	.	1.047...
8	114.143 820 712	.	.	.	1.019...
9	113.892 418 432	.	.	.	1.001 451...
10	113.766 717 291	.	.	.	.989 159...
11	113.829 567 861	.	.	.	.995 768...
12	113.860 993 147	.	.	.	.998 702...
13	113.876 705 789	.	.	.	1.000 100...
	.	.	.	.	.
	.	.	.	.	.
	.	.	.	.	.
32	113.875 558 108	.578 621 1313	.035 458 7918	.385 920 0779	1.000 000 0010
33	113.875 558 093	.578 621 1312	.035 458 7907	.385 920 0778	.999 999 9997
34	113.875 558 100	.578 621 1313	.035 458 7912	.385 920 0778	1.000 000 0003
35	113.875 558 096	.578 621 1312	.035 458 7910	.385 920 0778	1.000 000 0000

We begin with $\alpha = \mu_2$ since that is the largest μ_i—any smaller positive value of α would make R_2 imaginary. (Incidentally, we see here the purpose of the absolute-value signs in the flow-chart definition for R_i: although $\alpha = \mu_2$ makes $R_2 = 0$, in practice a round-off error can make the radicand negative.) At the end of Step 1, ΣR_i is <1, so we must hereafter use the same sign with R_2 as with R_1 and R_3. (If that first ΣR_i had been >1, we would thereafter have to use the opposite sign.) For Step 2, we use the average between $113.641 \cdots$, which is too small, and $2N = 178$, which is too large. At the end of Steps 2, 3, 4, 5, 6, 7, 8, and 9, ΣR_i is >1, so for the next α in each case we go half-way back to the last α making $\Sigma R_i < 1$—namely $113.641 \cdots$ At the end of Steps 10, 11 and 12, ΣR_i is <1, so in each case we average the current α with the last one making $\Sigma R_i > 1$—namely $113.892 \cdots$. At the end of Step 13, ΣR_i is >1 so we go half-way back to the last α, $113.860 \cdots$. We continue in this fashion till we find α correct to the desired degree of accuracy.

(The process can be speeded up considerably if, rather than always choosing the midpoint of the interval remaining, we use linear interpolation at each step. This is of some importance if the work is done on a desk calculator. Moreover, if the calculator contains a few storage registers, we can dispense with writing down any intermediate results.)

From the final value of α, and the formulas

$$P_i = (1/2)\left(1 + \frac{n_i - m_i}{\alpha} - R_i\right) \quad \text{and} \quad Q_i = (1/2)\left(1 + \frac{m_i - n_i}{\alpha} - R_i\right),$$

we have

$$P_1 = .114\ 092\ 7618 \qquad Q_1 = .307\ 286\ 1070$$
$$P_2 = .794\ 014\ 4115 \qquad Q_2 = .170\ 526\ 7975$$
$$P_3 = .091\ 892\ 8267 \qquad Q_3 = .522\ 187\ 0955$$
$$\text{and } t = .781\ 554\ 8963.$$

(Of course, we have given far more significant figures than are warranted by a sample size of 89.) Then, using the formulas

$$E_{ij} = p_{ij} \times 89 = \frac{P_i Q_j}{t} \times 89 \quad (i \neq j) \quad \text{and} \quad E_{ii} = 0,$$

we find the expected values

0	2.216	6.784
27.784	0	47.216
3.216	1.784	0

which, when compared with the observed data, give a χ^2 value of 2.257 ($df = K^2 - 3K + 1 = 1$), and, hence, a fiducial level of 86.7 percent ($P = .133$). In other words, the discrepancy between the observed and the expected values is not significant, and the data are adequately explained by the null hypothesis.

It is interesting to note that, if one makes the predictions on the basis of the naive approximations $P_i = n_i/89$ and $Q_i = m_i/89$, he will find a deviation from observed which is very highly significant ($P < .001$), and may be led (incorrectly) to reject the hypothesis of sender-receiver independence. This of course illustrates the general principle that a model can be rejected only on the basis of the best possible values of its parameters.

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